

ADDITIVE POSETS, CW-COMPLEXES, AND GRAPHS

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ABSTRACT. We introduce additive posets and study their properties and invariants. We show that the top homology group (with coefficients in $\mathbb{Z}/2\mathbb{Z}$) of a finite dimensional CW-complex carries a natural structure of an additive poset invariant under subdivisions of the CW-complex. Applications to graphs are discussed.

1. INTRODUCTION

Working on dimer coverings of graphs, see [Tu], the author was puzzled by the following elementary question: which 1-dimensional homology classes of a graph can be realized by embedded circles? For example, if the graph is a wedge of two or more circles, then these circles are embedded in the graph and represent generators of its first homology group. All other elements of this group cannot be represented by embedded circles. An analysis of this phenomenon leads to an algebraic formalism of additive posets. We define additive posets and discuss their properties and invariants. We also define certain elements of additive posets called atoms and tiles.

Additive posets naturally arise in the study of CW-complexes. For a CW-complex X of finite dimension $n \geq 0$, each element of the homology group $H = H_n(X; \mathbb{Z}/2\mathbb{Z})$ is uniquely represented by an n -cycle, that is by a finite collection of n -cells of X such that every $(n-1)$ -cell of X is incident to an even number of n -cells in this collection (counting with multiplicities). We say that a homology class $a \in H$ is smaller than or equal to a homology class $b \in H$ if the collection of n -cells representing a is contained in the collection of n -cells representing b . This partial order turns H into an additive poset preserved under subdivisions of X . We call H the homological additive poset of X . We prove that for $n \geq 2$, every finite additive poset is realizable as the homological additive poset of a finite n -dimensional CW-complex. A similar claim for $n = 1$, i.e., for graphs, does not hold: some finite additive posets are not realizable as homological additive posets of graphs.

As a geometric application, we estimate the number of n -cells of an n -dimensional CW-complex X from below in terms of the partial order in $H = H_n(X; \mathbb{Z}/2\mathbb{Z})$. This improves the standard estimate (the number of n -cells) $\geq \dim_{\mathbb{Z}/2\mathbb{Z}} H$. We also consider the problem of realization of elements of H by embeddings of closed n -manifolds into X . We show that only tiles of H can be realized by such embeddings. Only atoms of H can be realized by embeddings of closed connected n -manifolds into X . For $n = 1$, we have a more precise statement: a 1-dimensional homology class of a graph can be realized by an embedded circle if and only if this homology class is an atom.

The paper ends with a list of open problems.

¹AMS Subject Classification: 05C38, 06A11, 57Q05

2. ADDITIVE POSETS AND THEIR MORPHISMS

2.1. Additive posets. Recall that a (non-strict) partial order in a set P is a binary relation \leq over P such that for all $a, b, c \in P$, we have: $a \leq a$ (reflexivity); if $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetry); if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity). A set endowed with a partial order is called a partially ordered set or a poset.

By an *additive poset* we mean a pair (an abelian group A , a partial order \leq in A) such that for any $a, b, c \in A$, the following two conditions are met:

- (*) if $b \leq a$ and $c \leq a$, then $b + c \leq a$;
- (**) if $a \leq b$ and $a \leq c$, then $a \leq a + b + c$.

We will usually denote an additive poset (A, \leq) simply by A .

Lemma 2.1. *Let A be an additive poset and let $0 \in A$ be the zero element. Then for any $a \in A$, we have $a + a = 0$ and $0 \leq a$.*

Proof. By the definition of a partial order, $a \leq a$. Condition (*) implies that $a + a \leq a$. A second application of (*) yields that $a + a + a \leq a$. Also, an application of Condition (**) to the relation $a \leq a$ yields $a \leq a + a + a$. These two facts imply that $a + a + a = a$. Consequently, $a + a = 0$ and $0 = a + a \leq a$. \square

Lemma 2.1 shows that all elements of an additive poset have order two. This allows us to treat additive posets as vector spaces over the field $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. Lemma 2.1 implies that the zero vector is the (unique) least element of any additive poset. We will often use the following corollary of this fact: if a vector a in an additive poset satisfies $a \leq 0$, then $a = 0$.

The partial order in an additive poset A restricted to any subgroup of A turns the latter into an additive poset. It is called an additive subposet of A . In particular, for any $a \in A$, the set $\{b \in A \mid b \leq a\}$ is a subgroup of A as follows directly from Condition (*) above and the second claim of Lemma 2.1. This subgroup, viewed as an additive subposet of A , is denoted by A_a and called the *tail* of a . Clearly, $a \in A_a$ and $A_b \subset A_a$ for any $b \leq a$. Also, $A_0 = \{0\}$.

2.2. Examples. 1. Any $\mathbb{Z}/2\mathbb{Z}$ -vector space A carries a partial order \leq_t defined by $0 \leq_t a \leq_t a$ for all $a \in A$ (and no other relations). The pair (A, \leq_t) is an additive poset. We call \leq_t the *trivial partial order* in A .

2. Each set I gives rise to an additive poset 2^I whose elements are arbitrary subsets of I . The partial order in 2^I is given by the inclusion: $X \leq Y$ if $X \subset Y$ for $X, Y \in 2^I$. Addition in 2^I is the symmetric difference: $X + Y = (X \setminus Y) \cup (Y \setminus X)$ for $X, Y \in 2^I$. The zero element of 2^I is the empty set. All properties of an additive poset are straightforward. We call 2^I the *additive powerset* of I .

The additive powerset 2^I can be described in terms of maps $I \rightarrow \mathbb{Z}/2\mathbb{Z}$. Such maps bijectively correspond to subsets of I by assigning to every set $X \subset I$ its characteristic map $I \rightarrow \mathbb{Z}/2\mathbb{Z}$ carrying all elements of X to 1 and all elements of $I \setminus X$ to 0. Addition in 2^I corresponds in this language to the addition of functions. The partial order in 2^I is formulated in this language as follows: a map $a : I \rightarrow \mathbb{Z}/2\mathbb{Z}$ is smaller than or equal to a map $b : I \rightarrow \mathbb{Z}/2\mathbb{Z}$ if and only if $a(i) \leq_t b(i)$ for all $i \in I$, where \leq_t is the trivial partial order in $\mathbb{Z}/2\mathbb{Z}$.

3. Finite subsets of a set I form an additive subposet of 2^I denoted 2_f^I and called the *restricted additive powerset*. The additive poset 2_f^I can be equivalently formulated in terms of the $\mathbb{Z}/2\mathbb{Z}$ -vector space

$$A = \oplus_{i \in I} (\mathbb{Z}/2\mathbb{Z}) i.$$

The elements of A can be identified with finite subsets of I by assigning to each finite set $X \subset I$ the vector $\sum_{i \in X} i \in A$. Under this identification, addition in 2_f^I corresponds to addition in A , and the partial order in 2_f^I corresponds to the following partial order in A : a vector $a = \sum_{i \in I} a_i i$ is smaller than or equal to a vector $b = \sum_{i \in I} b_i i$ (with $a_i, b_i \in \mathbb{Z}/2\mathbb{Z}$ for all i) if and only if $a_i \leq_t b_i$ for all $i \in I$. If I is a finite set, then $2^I = 2_f^I$.

4. Given a set I , the finite subsets of I with an even number of elements form an additive subposet of 2_f^I denoted by 2_{ev}^I .

5. Let R be a Boolean ring that is a ring such that $x^2 = x$ for all $x \in R$. It is known that such a ring is commutative and $x + x = 0$ for all $x \in R$. The canonical partial order in R is defined by $x \leq y$ if $x = xy$. The underlying abelian group of R with this partial order is an additive poset.

2.3. Morphisms of additive posets. A *morphism* of additive posets $A \rightarrow B$ is a group homomorphism $\varphi : A \rightarrow B$ which is *order-preserving* in the sense that for all $a, b \in A$ satisfying $a \leq b$, we have $\varphi(a) \leq \varphi(b)$. Additive posets and their morphisms form a category with the obvious composition of morphisms. An isomorphism in this category is a bijective map between additive posets which is both a group isomorphism and a poset isomorphism.

For example, a map g from a set I to a set J induces a morphism of additive posets $g^* : 2^J \rightarrow 2^I$ carrying each subset of J to its pre-image under g . If g is injective, then it induces a morphism of additive posets $g_* : 2^I \rightarrow 2^J$ carrying each subset of I to its image under g . If g is a bijection, then g^* and g_* are mutually inverse isomorphisms of additive posets.

2.4. Remarks. 1. The partial order in a nonzero additive poset A is not translation invariant. Indeed, for any nonzero $a \in A$ we have $0 \leq a$. If the order is translation invariant, then $a = a + 0 \leq a + a = 0$, i.e., $a = 0$, a contradiction.

2. Three elements a, b, c of an additive poset satisfy the relations $a \leq b$, $a \leq c$, and $a \leq b + c$ if and only if $a = 0$. The “if” part is obvious. The “only if” part: by (**), the first two relations imply that $a \leq a + b + c$. If $a \leq b + c$, then another application of (**) gives

$$a \leq a + (a + b + c) + (b + c) = (a + a) + (b + b) + (c + c) = 0.$$

Consequently, $a = 0$.

3. Given an additive poset A , consider the partial binary operation \setminus on A defined by $a \setminus b = a + b$ for $a, b \in A$ such that $b \leq a$. This is a difference operation in the sense of [KC]. If A has a greatest element, then the underlying poset of A with the operation \setminus is a difference poset in the sense of [NP, KC].

3. A CONSTRUCTION OF ADDITIVE POSETS

We describe a general construction producing additive posets from families of linear functionals on $\mathbb{Z}/2\mathbb{Z}$ -vector spaces.

3.1. Construction. We start with a lemma which derives an additive poset from another additive poset.

Lemma 3.1. *Let A be a $\mathbb{Z}/2\mathbb{Z}$ -vector space and let (B, \leq) be an additive poset. Any set S of linear maps from A to B determines a binary relation \preceq in A as follows:*

$a \preceq b$ for $a, b \in A$ if (and only if) $s(a) \leq s(b)$ for all $s \in S$. If $\bigcap_{s \in S} \text{Ker } s = 0$, then (A, \preceq) is an additive poset.

Proof. Reflexivity and transitivity of \preceq follow from the corresponding properties of \leq . To prove the antisymmetry of \preceq , consider any $a, b \in A$ such that $a \preceq b$ and $b \preceq a$. Then $s(a) \leq s(b)$ and $s(b) \leq s(a)$ for all $s \in S$. Since \leq is a partial order, $s(a) = s(b)$ for all $s \in S$. Therefore $a - b \in \bigcap_{s \in S} \text{Ker } s = 0$ so that $a = b$.

We verify Condition (*) in the definition of an additive poset. If $a, b, c \in A$ satisfy $b \preceq a$ and $c \preceq a$, then for any $s \in S$, we have $s(b) \leq s(a)$ and $s(c) \leq s(a)$. Since s is an additive map and (B, \leq) is an additive poset,

$$s(b + c) = s(b) + s(c) \leq s(a).$$

Consequently $b + c \preceq a$. Condition (**) is verified similarly: if $a, b, c \in A$ satisfy $a \preceq b$ and $a \preceq c$, then for all $s \in S$, we have $s(a) \leq s(b)$ and $s(a) \leq s(c)$. Since s is an additive map and (B, \leq) is an additive poset,

$$s(a) \leq s(a) + s(b) + s(c) = s(a + b + c).$$

Consequently, $a \preceq a + b + c$. □

For $B = \mathbb{Z}/2\mathbb{Z}$ with the trivial partial order \leq_t , Lemma 3.1 yields the following theorem.

Theorem 3.2. *Let A be a $\mathbb{Z}/2\mathbb{Z}$ -vector space and $A^* = \text{Hom}(A, \mathbb{Z}/2\mathbb{Z})$. Any set $S \subset A^*$ determines a binary relation \preceq in A as follows: $a \preceq b$ for $a, b \in A$ if (and only if) $s(a) \leq_t s(b)$ for all $s \in S$. If $\bigcap_{s \in S} \text{Ker } s = 0$, then the pair (A, \preceq) is an additive poset.*

To ensure the condition $\bigcap_{s \in S} \text{Ker } s = 0$ in Theorem 3.2, it suffices to require that S generates A^* as a vector space.

3.2. Examples. In the following examples, A is a $\mathbb{Z}/2\mathbb{Z}$ -vector space.

1. For $S = A^*$, Theorem 3.2 yields the trivial partial order in A . Indeed, for any distinct nonzero vectors $a, b \in A$, there is a homomorphism $s : A \rightarrow \mathbb{Z}/2\mathbb{Z}$ such that $s(a) = 1$ and $s(b) = 0$. This shows that $a \not\preceq b$ and leaves only the relations $0 \leq a \leq a$ for all $a \in A$.

2. For a nonzero vector $a_0 \in A$, consider the set $S \subset A^*$ consisting of all homomorphisms $A \rightarrow \mathbb{Z}/2\mathbb{Z}$ which carry a_0 to 1. Applying Theorem 3.2 we obtain a partial order in A which turns A into an additive poset. This partial order may be directly defined by $0 \leq a \leq a_0$ for all $a \in A$.

3. For a $\mathbb{Z}/2\mathbb{Z}$ -vector subspace H of A and a nonzero vector $a_0 \in H$, consider the set $S \subset A^*$ consisting of all homomorphisms $A \rightarrow \mathbb{Z}/2\mathbb{Z}$ which either carry a_0 to 1 or carry H to 0. Applying Theorem 3.2 we obtain a partial order in A which turns A into an additive poset. This partial order may be directly defined by $0 \leq a \leq a_0$ for all $a \in A$ and $a \leq a_0$ for all $a \in H$. For $H = A$, we obtain the previous example.

4. INDEPENDENCE AND CHAINS

We introduce a partial relation in additive posets called independence.

4.1. Independence. Elements a and b of an additive poset A are said to be *independent* (from each other) if $a \leq a + b$. Since $a + b \leq a + b$, the first claim of Lemma 2.1 and Condition (*) in Section 2.1 imply that for independent a, b ,

$$b = a + (a + b) \leq a + b = b + a.$$

Consequently, the relation of independence is symmetric. Any independent nonzero vectors $a, b \in A$ are incomparable in the sense that neither $a \leq b$ nor $b \leq a$. Indeed, if, for example, $a \leq b$, then applying (**) to the relations $a \leq b$ and $a \leq a + b$, we obtain that $a \leq 0$, which contradicts the assumption $a \neq 0$.

Lemma 4.1. *Let a be an element of an additive poset A and let $A^a \subset A$ be the set of all elements of A independent from a . Then A^a is a subgroup of A such that $A^a \cap A_a = \{0\}$. The tail of any element of A^a is contained in A^a .*

Proof. The definition of independence implies that $0 \in A^a$. For any $b, c \in A^a$, we have $a \leq a + b$ and $a \leq a + c$. Condition (**) from Section 2.1 implies that

$$a \leq a + (a + b) + (a + c) = a + b + c.$$

Thus, $b + c \in A^a$. We conclude that A^a is a subgroup of A . The equality $A^a \cap A_a = \{0\}$ holds because all nonzero elements of A^a are incomparable with a .

To prove the last claim of the lemma, pick $b \in A^a$ and pick an element c of the tail of b . Then $c \leq b \leq a + b$, and therefore $c \leq a + b$. Applying Condition (**) from Section 2.1 to the relations $c \leq b$ and $c \leq a + b$, we obtain that

$$c \leq c + b + (a + b) = a + c.$$

Thus, $c \in A^a$. □

Lemma 4.2. *Any set of pairwise independent nonzero vectors in an additive poset is linearly independent.*

Proof. If the set in question is not linearly independent, then it has a finite subset, say, $\{b_1, \dots, b_n\}$ such that $b_1 + \dots + b_n = 0$. For $n = 1$, this contradicts the assumption $b_1 \neq 0$. Suppose that $n \geq 2$. Since the vectors b_2, \dots, b_n are independent from b_1 , their sum is also independent from b_1 , see Lemma 4.1. Then

$$b_1 \leq b_1 + (b_2 + \dots + b_n) = 0.$$

Consequently, $b_1 = 0$ which contradicts the assumptions of the lemma. □

4.2. Chains. For elements a, b of a poset P , one writes $a < b$ if $a \leq b$ and $a \neq b$. A *chain of length $n \geq 1$* in P is a sequence $a_0, a_1, \dots, a_n \in P$ such that $a_0 < a_1 < \dots < a_n$.

Lemma 4.3. *Let A be an additive poset. For any $a \in A$ and any integer $n \geq 1$, there is a bijective correspondence between chains of length n in A starting with a and sequences of n pairwise independent nonzero vectors in A^a . The correspondence carries a chain*

$$(4.2.1) \quad a = a_0 < a_1 < a_2 < \dots < a_n$$

into the sequence b_1, \dots, b_n where $b_i = a_{i-1} + a_i$ for $i = 1, \dots, n$. The inverse correspondence carries a sequence $b_1, \dots, b_n \in A$ into the chain (4.2.1) defined by $a_i = a + b_1 + \dots + b_i$ for $i = 1, \dots, n$.

Proof. Consider a chain (4.2.1) in A and set $b_i = a_{i-1} + a_i \in A$ for $i = 1, \dots, n$. We claim that b_1, \dots, b_n are pairwise independent nonzero vectors of A^a . That $b_i \neq 0$ for all i follows from the assumption $a_{i-1} < a_i$ which implies that $a_{i-1} \neq a_i$. Since $a_{i-1} \leq a_i$, Condition (*) in the definition of an additive poset implies that

$$b_i = a_{i-1} + a_i \leq a_i = a_{i-1} + b_i.$$

This proves that $b_i \leq a_i$ and that b_i is independent from a_{i-1} for all i . Now, for any $j = 1, \dots, i-1$, we have $a_j \leq a_{i-1}$. Since b_i is independent from a_{i-1} , Lemma 4.1 implies that b_i is independent from a_j . Since $b_j \leq a_j$, Lemma 4.1 implies that b_i is independent from b_j . Note also that b_i is independent from $a_0 = a$ so that $b_i \in A^a$. This proves our claim above.

Conversely, consider a sequence b_1, \dots, b_n of pairwise independent nonzero vectors in A^a . Set $a_0 = a$ and $a_i = a + b_1 + \dots + b_i$ for $i = 1, \dots, n$. This yields a chain (4.2.1) in A . Indeed, $a_{i-1} \neq a_i$ for all i because $b_i \neq 0$. By the assumptions, the vectors a, b_1, \dots, b_{i-1} are independent from b_i , and therefore, by Lemma 4.1 so is their sum a_{i-1} . Thus, $a_{i-1} \leq a_{i-1} + b_i = a_i$.

It follows from the definitions that the above correspondences are mutually inverse. This proves the claim of the lemma. \square

4.3. Example. For a set I , elements of the additive powerset 2^I are independent if and only if they are disjoint as subsets of I . The same holds for 2_f^I and 2_{ev}^I .

4.4. Exercise. Prove that for any independent elements a, b of an additive poset A , we have $A_{a+b} \supset A_a + A_b$ and $A^{a+b} = A^a \cap A^b$.

5. ATOMS AND TILES

5.1. Atoms. One says that an element a of a poset P covers an element $b \in P$ if $b < a$ and there is no $c \in P$ such that $b < c < a$. An element of an additive poset is an *atom* if it covers 0. In other words, an element a of an additive poset is an atom if (and only if) $a \neq 0$ and the tail A_a of a has no elements other than 0 and a .

Lemma 5.1. *An element a of an additive poset A covers an element $b \in A$ if and only if $a + b$ is an atom of A and $a + b \leq a$.*

Proof. Suppose that $a + b \in A_a$ is an atom. Then $a + b \neq 0$ and so $a \neq b$. By Condition (*), the relations $a + b \leq a$ and $a \leq a$ imply that $b \leq a$. Thus $b < a$. If there is $c \in A$ such that $b < c < a$, then applying Lemma 4.3 to the chain $b < c < a$ we obtain that $b + c$ and $c + a$ are independent nonzero vectors. Hence

$$0 < b + c < (b + c) + (c + a) = a + b.$$

This contradicts the assumption that $a + b$ is an atom. Thus, a covers b .

Conversely, suppose that a covers b . Then $b < a$ and, in particular, $a \neq b$. Therefore $a + b \neq 0$. By Condition (*), the relations $a \leq a$ and $b \leq a$ imply that $a + b \leq a$. If $a + b$ is not an atom, then there is $c \in A$ such that $0 < c < a + b$. Combining with $a + b \leq a$, we obtain that $c < a$. Using Condition (*) and the relations $b < a$ and $c < a$, we obtain that $b + c \leq a$. Since $c \neq a + b$, we have $b + c \neq a$. Thus $b + c < a$. Next, using Condition (**), we deduce from $c < a + b$ and $c < a$ that $c \leq b + c$. Thus, b and c are independent and consequently, $b \leq b + c$. Since $c \neq 0$, we have $b \neq b + c$. Thus, $b < b + c < a$. This contradicts the assumption that a covers b . Hence, $a + b$ is an atom. \square

Lemma 5.1 may be interpreted in terms of the Hasse diagrams. The Hasse diagram of a poset P is a directed graph with the set of vertices P which has an edge directed from $a \in P$ to $b \in P$ if and only if a covers b . By the Hasse diagram of an additive poset we mean the Hasse diagram of the underlying poset. Lemma 5.1 implies that the elements of an additive poset A covered by a vector $a \in A$ bijectively correspond to the atoms of A that are smaller than or equal to a . These atoms are nothing but the atoms of the additive poset $A_a \subset A$. The correspondence carries an element $b \in A$ covered by a to $a + b \in A_a$. The inverse correspondence carries an atom $c \in A_a$ to $a + c$. Similarly, the elements of A covering a vector $b \in A$ bijectively correspond to the atoms of A independent from b . These atoms are nothing but the atoms of the additive poset $A^b \subset A$. The correspondence carries an element $a \in A$ covering b to $a + b \in A^b$. The inverse correspondence carries an atom $c \in A^b$ to $b + c$.

5.2. Tiles. An element a of an additive poset A is a *tile* if $a \neq 0$ and any two distinct atoms of A belonging to the tail of a are independent from each other. Clearly, all atoms are tiles. All nonzero elements of the tail of a tile are tiles.

Theorem 5.2. *Let a be a tile of an additive poset A . Let I be the set of all atoms of the tail A_a of a . The additive subposet of A_a generated by I as a $\mathbb{Z}/2\mathbb{Z}$ -vector space is isomorphic to the restricted additive powerset 2_f^I .*

Proof. For each finite set $J \subset I$, put $\varphi(J) = \sum_{j \in J} j \in A_a$. This defines an additive homomorphism $\varphi : 2_f^I \rightarrow A_a$. It is injective as directly follows from the assumption that a is a tile and Lemma 4.2. We need only to verify that φ carries the partial order in 2_f^I determined by the inclusion of sets into the partial order in $\varphi(2_f^I)$ induced by that of A . Observe that if two finite sets $J, L \subset I$ are disjoint, then elements of J are independent from elements of L . By Lemma 4.1, the vectors $\varphi(J)$ and $\varphi(L)$ are independent from each other. Thus, $\varphi(J) \leq \varphi(J) + \varphi(L) = \varphi(J \cup L)$. Consequently, for any finite sets $J \subset K \subset I$, we have $\varphi(J) \leq \varphi(K)$. Conversely, suppose that $\varphi(J) \leq \varphi(K)$ for some finite sets $J, K \subset I$. For any $j_0 \in J$, we have the following relations in A :

$$j_0 \leq \sum_{j \in J} j = \varphi(J) \leq \varphi(K).$$

Therefore $j_0 \leq \varphi(K)$. If $j_0 \notin K$, then the vectors j_0 and $\varphi(K) = \sum_{k \in K} k$ are independent and therefore incomparable. This contradicts the relation $j_0 \leq \varphi(K)$. Thus, $j_0 \in K$. This proves that $J \subset K$. \square

5.3. Examples. 1. All nonzero vectors in an additive poset with trivial partial order are atoms.

2. For a set I , the atoms of the additive posets 2^I and 2_f^I are the one-element subsets of I . The atoms of 2_{ev}^I are the two-element subsets of I . All nonzero elements of 2_f^I are tiles. A nonzero element of 2^I is a tile if and only if it belongs to $2_f^I \subset 2^I$. The additive poset 2_{ev}^I has no tiles other than its atoms.

5.4. Remark. For a finite set I , the additive poset 2_{ev}^I is isomorphic to an additive powerset if and only if $|I| \leq 2$. Indeed, if $I = \emptyset$ or $|I| = 1$, then $2_{ev}^I = \{0\} = 2^\emptyset$. If $|I| = 2$, then $2_{ev}^I = 2^J$ where J is a one-element set. If $|I| = 3$ or, more generally, if $|I| \geq 3$ is odd, then 2_{ev}^I does not have a greatest element while any additive

powerset has a greatest element. If $|I| \geq 4$, then 2_{ev}^I has nonzero elements that are not tiles while all nonzero elements of a finite additive powerset are tiles.

6. FINITE ADDITIVE POSETS

An additive poset is *finite* if its underlying set is finite. In this section, we discuss properties of finite additive posets.

6.1. Atoms as generators. The following theorem shows that every element of a finite additive poset expands in a canonical way as a sum of atoms.

Theorem 6.1. *Let A be a finite additive poset and let $\mu : A \times A \rightarrow \mathbb{Z}$ be the Möbius function of the partial order in A . Then any vector $a \in A$ expands as $a = \sum_{b \in \mathcal{A}} \mu(b, a) b$ where $\mathcal{A} \subset A$ is the set of all atoms of A .*

Proof. The Möbius function μ is uniquely characterized by the following properties. For all $a \in A$, we have $\mu(a, a) = 1$. For all distinct $a, b \in A$, we have $\mu(a, b) = 0$ if $a \not\leq b$ and $\sum_{a \leq c \leq b} \mu(a, c) = 0$ if $a \leq b$. Given an arbitrary map f from A to an abelian group B , one defines a map $g : A \rightarrow B$ by

$$g(a) = \sum_{b \in A_a} f(b) \quad \text{for all } a \in A.$$

The Möbius inversion formula says that for all $a \in A$,

$$(6.1.1) \quad f(a) = \sum_{b \in A} \mu(b, a) g(b).$$

We apply these formulas to the identity map $f = \text{id} : A \rightarrow A$. The map g carries any $a \in A$ to $\sum_{b \in A_a} b$. If a is an atom, then $g(a) = 0 + a = a$. If a is not an atom, then $g(a) = 0$; this follows, for example, from the fact that the vector $\sum_{b \in A_a} b \in A_a$ is invariant under all automorphisms of the $\mathbb{Z}/2\mathbb{Z}$ -vector space A_a and $\dim_{\mathbb{Z}/2\mathbb{Z}} A_a \geq 2$. Now, Formula (6.1.1) directly implies the claim of the theorem. \square

Corollary 6.2. *Any finite additive poset A is generated as a $\mathbb{Z}/2\mathbb{Z}$ -vector space by its atoms.*

The following theorem produces a different kind of expansions of elements of a finite additive poset as a sum of atoms.

Theorem 6.3. *Every nonzero element a of a finite additive poset A expands as a sum of pairwise independent atoms of A . The atoms in any such expansion of a belong to the tail of a .*

Proof. One says that a chain $a_0 < a_1 < \dots < a_n$ in a poset is *saturated* if a_i covers a_{i-1} for $i = 1, \dots, n$. Lemma 5.1 implies that under the bijective correspondence of Lemma 4.3, saturated chains in A starting with $a_0 = 0$ correspond to sequences of pairwise independent atoms b_1, \dots, b_n in $A^0 = A$. The maximal element a_n of the chain is computed by $a_n = b_1 + \dots + b_n$. Since A is finite, for any $a \in A$ there is a saturated chain with maximal element a . For the corresponding pairwise independent atoms b_1, \dots, b_n , we have $a = b_1 + \dots + b_n$.

To prove the second claim of the theorem consider an expansion of a as a sum of pairwise independent atoms $a_1, \dots, a_n \in A$ with $n \geq 1$. If $n = 1$, then $a_1 = a \leq a$. If $n \geq 2$, then Lemma 4.1 implies that a_1 is independent from $b = a_2 + \dots + a_n$ so that $a_1 \leq a_1 + b = a$. Similarly, $a_i \leq a$ for all $i = 1, \dots, n$. \square

6.2. Remarks. 1. For any atom $a \in A$, the expansions of a in Theorems 6.1 and 6.3 are just $a = a$.

2. The expansion in Theorem 6.3 is not necessarily unique. For instance, given four distinct elements i, j, k, l of a set I , the element $\{i, j, k, l\}$ of the additive poset 2_{ev}^I expands as a sum of pairwise independent atoms in three ways:

$$\{i, j, k, l\} = \{i, j\} + \{k, l\} = \{i, k\} + \{j, l\} = \{i, l\} + \{j, k\}.$$

6.3. Tiles re-examined. We discuss the tiles of finite additive posets.

Theorem 6.4. *Let A be a finite additive poset. The following conditions on a nonzero vector $a \in A$ are equivalent:*

- (i) a is a tile;
- (ii) the tail A_a of a is isomorphic to an additive powerset;
- (iii) all atoms of A_a are pairwise independent and their sum is equal to a ;
- (iv) the expansion of a as a sum of pairwise independent atoms of A is unique up to permutation of the atoms.

Proof. The implication (i) \Rightarrow (ii) follows from Theorem 5.2 and the fact that the atoms of A_a additively generate A_a . Note also that the set I of atoms of A_a is finite and therefore $2_f^I = 2^I$.

We prove that (ii) \Rightarrow (iii). Under an identification of A_a with the additive powerset of a finite set I , the atoms of A_a correspond to singletons while the vector $a \in A_a$, being the maximal element of A_a , corresponds to the set I itself. It is clear that the singletons are pairwise disjoint and their union is I . This proves (iii).

We prove that (iii) \Rightarrow (iv). Given an expansion of a as a sum of pairwise independent atoms a_1, \dots, a_k of A , we have $a_1, \dots, a_k \in A_a$ by Theorem 6.3. We need only to show that all atoms of A_a appear in the sequence a_1, \dots, a_k . Let b_1, \dots, b_l be the atoms of A_a not appearing in the sequence a_1, \dots, a_k . By assumption, the atoms $a_1, \dots, a_k, b_1, \dots, b_l$ are pairwise independent and their sum is equal to a . Since $a = a_1 + \dots + a_k$, we have $b_1 + \dots + b_l = 0$. By Lemma 4.2, this means that $l = 0$.

We now prove that (iv) \Rightarrow (i). Suppose that a has a unique expansion as a sum of pairwise independent atoms a_1, \dots, a_n of A with $n \geq 1$. By Theorem 6.3, we have $a_1, \dots, a_n \in A_a$. If $c \in A_a$ is an atom, then either $a = c$ or $a \neq c$. In the former case, a is an atom and a tile. In the latter case, set $b = a + c \neq 0$. Since $c \leq a = c + b$, the atom c is independent from b . Pick an expansion of b as a sum of pairwise independent atoms $b_1, \dots, b_m \in A_b$ with $m \geq 1$. Lemma 4.1 implies that c is independent from b_j for all j . Thus, the atoms c, b_1, \dots, b_m are pairwise independent and their sum is equal to $c + b = a$. By assumption, the sets of atoms $\{a_1, \dots, a_n\}$ and $\{c, b_1, \dots, b_m\}$ must coincide. Thus, $c = a_i$ for some i . Therefore, all atoms of A_a belong to the set $\{a_1, \dots, a_n\}$. This implies that a is a tile. \square

6.4. Example. Let $a_0 \in H \subset A$ be as in Example 3.2.3 and $\dim(H) \geq 2$. Then $A_{a_0} = H$ and the set of atoms of A_{a_0} is $\mathcal{A} = H \setminus \{a_0, 0\}$. The expansion of a_0 in Theorem 6.1 is $a_0 = \sum_{a \in \mathcal{A}} a$. Theorem 6.3 yields an expansion of a_0 as a sum of two atoms b and c , where b is any element of \mathcal{A} and $c = a_0 + b \in \mathcal{A}$. The vector $a_0 \in A$ is a tile if and only if $\dim(A) = 2$.

7. INVARIANTS OF FINITE ADDITIVE POSETS

7.1. Invariants. We consider four numerical invariants of finite additive posets: the height, the width, the weight, and the dimension. The first two invariants are

defined for an arbitrary finite poset P . The *height* $h(P)$ is the maximal length of a chain in P . The *width* $w(P)$ of P is the cardinality of a biggest antichain of P where by an antichain one means a subset consisting of pairwise incomparable elements. The height $h(A)$ and width $w(A)$ of a finite additive poset A are the height and width of the underlying poset. The *weight* $wt(A)$ of A is the number of atoms of A . The *dimension* $\dim(A)$ of A is the dimension of A viewed as a $\mathbb{Z}/2\mathbb{Z}$ -vector space. Clearly, $\dim(A) = \log_2 |A|$ where the vertical bars denote the number of elements of a finite set.

Theorem 7.1. *For any finite additive poset A , we have*

$$h(A) \leq \dim(A) \leq wt(A) \leq w(A).$$

Proof. By definition, $n = h(A)$ is the maximal integer such that A has a chain of length n . Since $0 \in A$ is the least element of A , a chain of length n in A must start with 0 . Lemma 4.3 implies that n is the maximal integer such that there are n pairwise independent nonzero vectors in $A^0 = A$. By Lemma 4.2, $n \leq \dim(A)$. The inequalities $\dim(A) \leq wt(A) \leq w(A)$ follow from the fact that the atoms of A generate A as a vector space and form an antichain. \square

7.2. Example. For a finite set I ,

$$h(2^I) = \dim(2^I) = wt(2^I) = |I|.$$

By Sperner's theorem (see [En]), $w(2^I) = \binom{n}{\lfloor n/2 \rfloor}$, where $n = |I|$ and $\lfloor n/2 \rfloor \in \mathbb{Z}$ is the integral part of $n/2$.

7.3. The weight function. Let A be a finite additive poset. Any invariant ψ of finite additive posets determines two functions on A by

$$\psi(a) = \psi(A_a) \quad \text{and} \quad co - \psi(a) = \psi(A^a)$$

for all $a \in A$. In particular, we can apply these definitions to the invariants from Section 7.1. We briefly discuss the functions associated with $\psi = wt$.

We define the *weight* of $a \in A$ by $wt(a) = wt(A_a)$. Thus, $wt(a)$ is the number of atoms of A that are smaller than or equal to a . In other words, $wt(a)$ is the number of edges in the Hasse diagram of A directed from a to other vertices. It follows from Theorem 6.1 that the weight function is strictly increasing in the sense that if $a < b$, then $wt(a) < wt(b)$ for any $a, b \in A$.

We define the *coweight* of $a \in A$ by $cowt(a) = wt(A^a)$. Thus, $cowt(a)$ is the number of atoms of A independent from a or, equivalently, the number of edges of the Hasse diagram of A directed towards a . Since $A_a \cap A^a = \{0\}$, we have

$$wt(a) + cowt(a) \leq wt(A).$$

The properties of atoms imply that the coweight function is strictly decreasing: if $a < b$, then $cowt(a) > cowt(b)$ for any $a, b \in A$.

8. PLAIN ADDITIVE POSETS

We introduce a class of finite additive posets called plain additive posets.

8.1. Embeddings. An *embedding* of an additive poset A into an additive poset B is an isomorphism of A onto an additive subposet of B . In other words, an embedding of A into B is a group monomorphism $\varphi : A \rightarrow B$ such that for any $a, b \in A$, the relation $a \leq b$ holds in A if and only if the relation $\varphi(a) \leq \varphi(b)$ holds in B . We say that A *embeds* in B if there is an embedding $A \rightarrow B$.

We say that an additive poset A is *plain* if it embeds in the additive powerset 2^I for some finite set I . Then A is finite and

$$\dim(A) \leq \dim(2^I) = |I|.$$

8.2. Separating functionals. It is useful to reformulate plainness in terms of linear functionals. By a linear functional on an additive poset A , we mean a group homomorphism $A \rightarrow \mathbb{Z}/2\mathbb{Z}$, i.e., an element of the dual $\mathbb{Z}/2\mathbb{Z}$ -vector space

$$A^* = \text{Hom}(A, \mathbb{Z}/2\mathbb{Z}).$$

A linear functional $f : A \rightarrow \mathbb{Z}/2\mathbb{Z}$ is *order-preserving* if it is a morphism of additive posets $A \rightarrow (\mathbb{Z}/2\mathbb{Z}, \leq_t)$. In other words, f is order-preserving if for any $a, b \in A$ such that $a \leq b$, we have $f(a) \leq_t f(b)$.

We say that a set $S \subset A^*$ is *separating* if all elements of S are order-preserving and for any $a, b \in A$ with $a \not\leq b$, there is $s \in S$ such that $s(a) = 1$ and $s(b) = 0$. Applying the latter condition to $b = 0$ we obtain that for any nonzero $a \in A$, there is $s \in S$ such that $s(a) = 1$. In other words, $\bigcap_{s \in S} \text{Ker } s = 0$.

Theorem 8.1. *The following conditions on a finite additive poset A are equivalent:*

- (i) A is plain;
- (ii) The set of all order-preserving linear functionals on A is separating;
- (iii) There is a separating subset of A^* ;
- (iv) The partial order in A is obtained as in Theorem 3.2 from a set $S \subset A^*$ such that $\bigcap_{s \in S} \text{Ker } s = 0$.

Proof. We prove that (i) \Rightarrow (ii). Suppose that there is an embedding $\varphi : A \rightarrow 2^I$, where I is a finite set. For each $i \in I$, consider the map $s_i : 2^I \rightarrow \mathbb{Z}/2\mathbb{Z}$ which carries a set $X \subset I$ to 1 if $i \in X$ and to 0 if $i \notin X$. It is clear that s_i is an order-preserving linear map. Hence, $s_i \varphi : A \rightarrow \mathbb{Z}/2\mathbb{Z}$ is an order-preserving linear functional. To check (ii), pick any $a, b \in A$ with $a \not\leq b$. Then $\varphi(a) \not\leq \varphi(b)$, i.e., the set $\varphi(a) \subset I$ is not contained in the set $\varphi(b) \subset I$. For any $i \in \varphi(a) \setminus \varphi(b)$, we have $s_i \varphi(a) = 1$ and $s_i \varphi(b) = 0$.

The implication (ii) \Rightarrow (iii) is obvious. We show that (iii) \Rightarrow (iv). Let $S \subset A^*$ be a separating set. As we know, $\bigcap_{s \in S} \text{Ker } s = 0$. We claim that the given partial order \leq in A coincides with the partial order \preceq in A determined by S as in Theorem 3.2. Indeed, pick any $a, b \in A$. If $a \leq b$, then $s(a) \leq_t s(b)$ for all $s \in S$ because S consists of order-preserving linear functionals. By definition of \preceq , we have $a \preceq b$. If $a \not\leq b$, then, since S is separating, there is $s \in S$ such that $s(a) = 1$ and $s(b) = 0$. By definition of \preceq , we have $a \not\preceq b$. Therefore, $\leq = \preceq$.

We prove that (iv) \Rightarrow (i). Suppose that the partial order in A is obtained as in Theorem 3.2 from a set $S \subset A^*$ such that $\bigcap_{s \in S} \text{Ker } s = 0$. Consider the map $\varphi : A \rightarrow 2^S$ carrying any $a \in A$ to the set

$$\varphi(a) = \{s \in S \mid s(a) = 1\} \subset S.$$

The map φ is an additive homomorphism: for any $a, b \in A$,

$$\varphi(a + b) = \{s \in S \mid s(a + b) = 1\} = \{s \in S \mid s(a) + s(b) = 1\}$$

$$\begin{aligned}
&= \{s \in S \mid s(a) = 1, s(b) = 0 \text{ or } s(a) = 0, s(b) = 1\} \\
&= \{s \in S \mid s(a) = 1\} + \{s \in S \mid s(b) = 1\} = \varphi(a) + \varphi(b).
\end{aligned}$$

The condition $\bigcap_{s \in S} \text{Ker } s = 0$ ensures that if $a \neq 0$, then $\varphi(a) \neq \emptyset$. Thus, φ is a monomorphism. Next, consider any $a, b \in A$. If $a \leq b$, then $s(a) \leq_t s(b)$ for all $s \in S$. Thus, $s(a) = 1 \Rightarrow s(b) = 1$. Consequently, $\varphi(a) \leq \varphi(b)$. If $a \not\leq b$, then there is $s \in S$ such that $s(a) = 1$ and $s(b) = 0$. Then $s \in \varphi(a)$ and $s \notin \varphi(b)$. Consequently, $\varphi(a) \not\leq \varphi(b)$. We conclude that the map $\varphi : A \rightarrow 2^S$ is an embedding of additive posets. \square

8.3. Example. Any finite additive poset A with trivial partial order is plain. To see it, pick a basis V of A and denote by I the set of all subsets of V of cardinality 1 or 2. It is easy to check that the linear map $A \rightarrow 2^I$ carrying each vector $v \in V$ to the set $\{i \in I \mid v \in i\} \subset I$ is an embedding of additive posets.

9. COMPLEXITY OF PLAIN ADDITIVE POSETS

We introduce a numerical invariant of plain additive posets called complexity. Then we estimate the complexity from below in terms of the width.

9.1. Complexity. The *complexity* $c(A)$ of a plain additive poset A is the smallest integer $n \geq 0$ such that A embeds in the additive powerset 2^I for an n -element set I . If A is isomorphic to 2^I for a finite set I , then $c(A) = \dim(A) = |I|$. If A is not isomorphic to an additive powerset, then $c(A) \geq \dim(A) + 1$. If the partial order in A is obtained as in Theorem 3.2 from a set $S \subset A^*$ with $\bigcap_{s \in S} \text{Ker } s = 0$, then $c(A) \leq |S|$. This is clear from the proof of the implication $(iv) \Rightarrow (i)$ in Theorem 8.1.

The complexity is an isomorphism invariant of plain additive posets. It is monotonous in the sense that if an additive poset A embeds in a plain additive poset B , then A is plain and $c(A) \leq c(B)$.

Theorem 9.1. *For any plain additive poset A ,*

$$c(A) = \min_S |S|$$

where S runs over all separating subsets of A^ .*

Proof. An embedding $A \rightarrow 2^I$ determines a separating subset of A^* consisting of $|I|$ elements, see the proof of the implication $(i) \Rightarrow (ii)$ in Theorem 8.1. Therefore $\min_S |S| \leq c(A)$. Also, for any separating set $S \subset A^*$, there is an embedding $A \rightarrow 2^S$, see the proof of the implications $(iii) \Rightarrow (iv) \Rightarrow (i)$ in Theorem 8.1. Consequently, $c(A) \leq \min_S |S|$. \square

Theorem 9.2. *If $n = c(A)$ is the complexity of a plain additive poset A , then*

$$(9.1.1) \quad \binom{n}{\lfloor n/2 \rfloor} \geq w(A),$$

where $w(A)$ is the width of A .

Proof. Let I be an n -element set. By Sperner's theorem (see [En]), the number of elements in an antichain in 2^I cannot exceed $\binom{n}{\lfloor n/2 \rfloor}$. Since A embeds in 2^I , the same is true for antichains in A . This gives the inequality (9.1.1). \square

9.2. Examples. 1. For a finite set I , we compute the complexity of the additive poset 2_{ev}^I . If $I = \emptyset$ or $|I| = 1$, then $2_{ev}^I = 2^\emptyset$ and $c(2_{ev}^I) = \dim(2_{ev}^I) = 0$. If $|I| = 2$, then $2_{ev}^I = 2^J$, where J is a one-element set, and $c(2_{ev}^I) = \dim(2_{ev}^I) = 1$. Assume that $|I| \geq 3$ and pick $i \in I$. Set $J = I \setminus \{i\}$. The morphism of additive posets $2_{ev}^I \rightarrow 2^J, X \mapsto X \cap J$ is a bijection and hence a group isomorphism. (It is not an isomorphism of additive posets because for distinct $j, k \in J$, the elements $\{i, j\}$ and $\{j, k\}$ of 2_{ev}^I are incomparable in 2_{ev}^I while their images in 2^J are related by $\{j\} \leq \{j, k\}$.) Consequently, $\dim(2_{ev}^I) = |I| - 1$ and

$$|I| \geq c(2_{ev}^I) \geq \dim(2_{ev}^I) + 1 = |I|$$

where the left inequality holds because 2_{ev}^I is an additive subposet of 2^I and the right inequality holds because 2_{ev}^I is not isomorphic to an additive powerset (see Example 5.4). We conclude that $c(2_{ev}^I) = |I|$.

2. For an integer $m \geq 1$, let A_m be an m -dimensional $\mathbb{Z}/2\mathbb{Z}$ -vector space with the trivial partial order \leq_t . Clearly, $c(A_1) = 1$. Example 8.3 shows that

$$c(A_m) \leq m(m+1)/2.$$

By Theorem 9.2,

$$\binom{c(A_m)}{[c(A_m)/2]} \geq w(A_m) = 2^m - 1.$$

In particular, this implies that $c(A_2) = 3$ and gives the inequalities

$$6 \geq c(A_3) \geq 5, \quad 10 \geq c(A_4) \geq 6, \quad 15 \geq c(A_5) \geq 7.$$

We claim that $c(A_3) = 6$. Indeed, any separating set $S \subset A_3^*$ generates A_3^* and hence must contain a basis x, y, z of A_3^* . A direct verification shows that a set of vectors consisting of x, y, z and any two of other nonzero vectors

$$x + y, x + z, y + z, x + y + z$$

is non-separating. Hence $|S| \geq 6$ and $6 \geq c(A_3) = \min_S |S| \geq 6$.

10. HOMOLOGICAL ADDITIVE POSETS

We show that the top-dimensional homology of a CW-complex carries a natural structure of an additive poset. For basics on CW-complexes, see [LW].

10.1. CW-complexes. A k -ball with $k = 1, 2, \dots$ is a k -dimensional ball D^k in \mathbb{R}^k bounded by a $(k-1)$ -dimensional sphere S^{k-1} . To attach a k -ball to a topological space Y along a (continuous) map $\varphi : S^{k-1} \rightarrow Y$, one takes the disjoint union $Y \amalg D^k$ and identifies all points of $\partial D^k = S^{k-1}$ with their images under φ . One similarly attaches families of disjoint k -balls to Y along maps of their boundary spheres to Y . For an integer $n \geq 0$, an n -dimensional CW-complex is a Hausdorff topological space X endowed with a filtration

$$X^{(0)} \subset X^{(1)} \subset \dots \subset X^{(n)} = X,$$

where $X^{(0)}$ is a discrete set of points and for $k = 1, \dots, n$, the space $X^{(k)}$ is obtained by attaching a family of k -balls to $X^{(k-1)}$. The space $X^{(k)}$ is called the k -skeleton of X . The space $X^{(k)} \setminus X^{(k-1)}$ splits as a disjoint union of copies of $\text{Int}(D^k) = D^k \setminus S^{k-1}$ called the (open) k -cells of X . Denote the set of all k -cells of X by I_k . For $e \in I_k$ and $e' \in I_{k-1}$, one defines a residue $[e : e'] \in \mathbb{Z}/2\mathbb{Z}$ as follows. Compose the attaching map $S^{k-1} \rightarrow X^{(k-1)}$ of e with the map $X^{(k-1)} \rightarrow S^{k-1}$ obtained by collapsing $X^{(k-1)} \setminus e'$ to a point and identifying the resulting quotient

space of $X^{(k-1)}$ with S^{k-1} via an arbitrary homeomorphism. Then $[e : e']$ is the degree modulo 2 of the composed map $S^{k-1} \rightarrow S^{k-1}$.

Consider the *cellular chain complex* $C(X) = (\cdots \rightarrow C_k \rightarrow C_{k-1} \rightarrow \cdots)$ of X , where C_k is the $\mathbb{Z}/2\mathbb{Z}$ -vector space with basis I_k and the boundary homomorphism $\partial_k : C_k \rightarrow C_{k-1}$ carries each $e \in I_k$ to $\sum_{e' \in I_{k-1}} [e : e'] e'$. Note that the latter sum is finite because the closure of e meets only finitely many cells. For $k \geq 0$, the k -th *cellular homology group* of X with coefficients in $\mathbb{Z}/2\mathbb{Z}$ is the group

$$H_k(X; \mathbb{Z}/2\mathbb{Z}) = H_k(C(X)) = \text{Ker } \partial_k / \text{Im } \partial_{k+1}.$$

Clearly, $H_k(X; \mathbb{Z}/2\mathbb{Z}) = C_k = 0$ for all $k > n$. Set

$$H = H_n(X; \mathbb{Z}/2\mathbb{Z}) = \text{Ker } \partial_n \subset C_n.$$

Elements of C_n can be identified with finite subsets of I_n . A finite set $E \subset I_n$ corresponds to an element of H if and only if $\sum_{e \in E} [e : e'] = 0$ for all $e' \in I_{n-1}$. Such a finite set is called a *(cellular) n -cycle*. As we know, the partial order in $C_n = 2^{I_n}$ determined by the inclusion of subsets of I_n turns C_n into an additive poset. The restriction of this partial order to $H \subset C_n = 2^{I_n}$ turns H into a plain additive poset. We call H the *homological additive poset* of X . Clearly, the complexity of H is smaller than or equal to the number of n -cells of X :

$$(10.1.1) \quad c(H) \leq |I_n|.$$

The homological additive poset of X is invariant under subdivisions of X . A CW-complex Y is a *subdivision* of X if X and Y have the same underlying topological space and $X^{(k)} \subset Y^{(k)}$ for all $k \geq 0$. Then there is a chain homomorphism $C(X) \rightarrow C(Y)$ which carries a basis vector represented by a k -cell e of X to the sum of the basis vectors represented by the k -cells of Y contained in e . This homomorphism is a chain homotopy equivalence. Consequently, it induces an isomorphism $H_k(X; \mathbb{Z}/2\mathbb{Z}) \simeq H_k(Y; \mathbb{Z}/2\mathbb{Z})$ for all k . For $k = n = \dim(X) = \dim(Y)$, the latter isomorphism is an isomorphism of additive posets. This follows from the definitions and the following fact: if a cellular n -cycle in Y includes an n -cell of Y contained in an n -cell e of X , then this n -cycle includes all n -cells of Y contained in e .

10.2. Representation of homology classes. We relate the partial order above to the problem of representation of homology classes by manifolds. For $n \geq 0$, by a closed n -manifold we mean a non-empty compact n -dimensional topological manifold with empty boundary. Given a closed n -manifold M , we let $[M] \in H_n(M; \mathbb{Z}/2\mathbb{Z})$ be its fundamental class. A homology class $a \in H_n(X; \mathbb{Z}/2\mathbb{Z})$ of a CW-complex X is *represented* by a (continuous) map $i : M \rightarrow X$ if $i_*([M]) = a$.

Theorem 10.1. *Let X be an n -dimensional CW-complex with $n \geq 0$. Let $a, b \in H_n(X; \mathbb{Z}/2\mathbb{Z})$ be homology classes represented respectively by maps $i : M \rightarrow X$ and $j : N \rightarrow X$, where M, N are closed n -manifolds. If $i(M) \cap j(N) = \emptyset$, then a and b are independent.*

Proof. Since M and N are compact, so are their images $i(M) \subset X$ and $j(N) \subset X$. Since X is Hausdorff, both $i(M)$ and $j(N)$ are closed. If they are disjoint, then they have disjoint open neighborhoods $i(M) \subset U$ and $j(N) \subset V$. Taking a sufficiently small subdivision of X , we can assume that all n -cells of X meeting $i(M)$ are contained in U and all n -cells of X meeting $j(N)$ are contained in V . This implies that the homology classes $a = i_*([M])$ and $b = j_*([N])$ are represented by disjoint cellular n -cycles. The union of these n -cycles represents $a + b$. By the definition

of the partial order in $H_n(X; \mathbb{Z}/2\mathbb{Z})$, we have $a \leq a + b$, so that a and b are independent. \square

An *embedding* of a closed manifold M into a CW-complex X is an injective (continuous) map $M \rightarrow X$. Since M is compact and X is Hausdorff, such a map is a homeomorphism onto its image.

Theorem 10.2. *Let X be an n -dimensional CW-complex with $n \geq 0$. If a homology class $a \in H = H_n(X; \mathbb{Z}/2\mathbb{Z})$ is represented by an embedding of a closed n -manifold M into X , then a is a tile. Moreover, if M is connected, then a is an atom.*

Proof. Let $i : M \rightarrow X$ be an embedding representing a . As in the proof of Theorem 10.1, the set $i(M)$ is closed in X . Therefore, for any (open) n -cell e of X , the set $e \cap i(M)$ is closed in e . It is also open in e , as directly follows from the assumption that i is an embedding. Since e is connected, we have either $e \cap i(M) = \emptyset$ or $e \subset i(M)$. It is clear that the n -cells of X contained in $i(M)$ form an n -cycle representing the homology class $a = i_*([M]) \in H$. Since $M \neq \emptyset$ and i is an embedding, this n -cycle is not void, and so $a \neq 0$.

We prove next that if M is connected, then a is an atom. It suffices to show that any $b \in H$ such that $b < a$ is equal to 0. Such a b is represented by an n -cycle formed by some (but not all) n -cells of X contained in $i(M)$. These n -cells are contained in $i(M \setminus \{x\})$ for some point $x \in M$. Therefore, b lies in the image of the homomorphism

$$H_n(M \setminus \{x\}; \mathbb{Z}/2\mathbb{Z}) \rightarrow H = H_n(X; \mathbb{Z}/2\mathbb{Z})$$

induced by the restriction of i to $M \setminus \{x\}$. Since M is a connected n -dimensional manifold, $H_n(M \setminus \{x\}; \mathbb{Z}/2\mathbb{Z}) = 0$. Therefore $b = 0$.

Suppose now that M has $m \geq 2$ connected components M_1, \dots, M_m . For $k = 1, \dots, m$, set $a_k = i_*([M_k]) \in H$. Then

$$a = i_*([M]) = i_*([M_1] + \dots + [M_m]) = a_1 + \dots + a_m.$$

By Theorem 10.1 and the previous paragraph, this is an expansion of a as a sum of pairwise independent atoms. To prove that a is a tile, it is enough to show that every atom $b \leq a$ coincides with one of the atoms a_1, \dots, a_m . By the above, $a = i_*([M])$ is represented by the n -cycle E consisting of all n -cells of X contained in $i(M)$. The atom b is then represented by an n -cycle $F \subset E$. The set F splits as a disjoint union $\coprod_{k=1}^m F_k$ where F_k is the set of n -cells of X which belong to F and are contained in $i(M_k)$. Since the sets $\{i(M_k)\}_{k=1}^m$ are closed and pairwise disjoint, the assumption that F is an n -cycle implies that F_k is an n -cycle for all k . The arguments above show that if F_k includes all n -cells of X contained in $i(M_k)$, then F_k represents $a_k \in H$. Otherwise, F_k represents $0 \in H$. We conclude that the homology class b represented by F is a sum of several homology classes a_k . Since these classes are independent, they are smaller than or equal to their sum b . Since b is an atom, this may happen only when $b = a_k$ for some k . \square

10.3. Examples. 1. Let X be a wedge of several n -dimensional spheres with $n \geq 1$. The homology classes of these spheres form a basis, I , of $H_n(X; \mathbb{Z}/2\mathbb{Z})$. It is clear from the definitions that the homology additive poset of X is nothing but 2_f^I .

2. Consider a CW-complex Y obtained by gluing several n -balls along their boundary spheres. The homology additive poset of Y is nothing but 2_{ev}^I , where I is the set of n -balls forming Y .

10.4. Remarks. 1. The partial order in the homology additive poset is not, generally speaking, preserved under homotopy equivalences of CW-complexes. Consider, for instance, the CW-complex Y obtained by gluing three n -balls along their boundary spheres. Then $H_n(Y; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with trivial partial order. Collapsing one of the balls into a point, we turn Y into a CW-complex X which is homotopy equivalent to Y and is a wedge of two n -spheres. These n -spheres represent generators a, b of $H_n(X; \mathbb{Z}/2\mathbb{Z}) = H_n(Y; \mathbb{Z}/2\mathbb{Z})$. The partial order in $H_n(X; \mathbb{Z}/2\mathbb{Z})$ is non-trivial because $a \leq a + b$ and $b \leq a + b$.

2. The definition of the homology additive poset of a CW-complex readily extends to CW-pairs and relative CW-complexes. We do not pursue this line here.

11. REALIZATION OF ADDITIVE POSETS BY CW-COMPLEXES

11.1. Realization. A CW-complex is said to be *finite* if it has a finite number of cells. We prove the following theorem.

Theorem 11.1. *For any plain additive poset A and any integer $n \geq 2$, there is a finite n -dimensional CW-complex whose homological additive poset is isomorphic to A .*

The proof given below uses cohomology rather than homology, and we first introduce relevant notation. In the rest of the section, we fix an integer $n \geq 2$.

11.2. Cohomological computation of the partial order. Consider an n -dimensional CW-complex X . Each (open) n -cell e of X gives rise to a homotopy class of maps $s_e : X \rightarrow S^n$ obtained by collapsing $X \setminus e \subset X$ to a point and identifying the resulting quotient space of X with the sphere S^n . Set

$$X_e = (s_e)^*(z) \in H^n(X; \mathbb{Z}/2\mathbb{Z}),$$

where z is the nonzero element of $H^n(S^n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Let $S_X \subset H^n(X; \mathbb{Z}/2\mathbb{Z})$ be the set of cohomology classes associated in this way with the n -cells of X . Note that different n -cells of X may give rise to the same cohomology classes.

Consider next the $\mathbb{Z}/2\mathbb{Z}$ -vector space $H = H_n(X; \mathbb{Z}/2\mathbb{Z})$. Using the standard evaluation of cohomology classes on homology classes, we identify $H^n(X; \mathbb{Z}/2\mathbb{Z})$ with $H^* = \text{Hom}(H, \mathbb{Z}/2\mathbb{Z})$. Note that $\bigcap_{s \in S_X} \text{Ker } s = 0$. Indeed, any nonzero $a \in H$ is represented by a non-empty cellular n -cycle E in X , and then $X_e(a) = 1$ for all $e \in E$. By Theorem 3.2, the set S_X determines a partial order in H turning H into an additive poset. It follows from the definitions that this partial order in H coincides with the partial order defined in Section 10.1 via inclusions of n -cycles.

11.3. Realizable pairs. We say that a pair (a $\mathbb{Z}/2\mathbb{Z}$ -vector space B , a set $S \subset B$) is *realized* by an n -dimensional CW-complex X if there is an isomorphism of $\mathbb{Z}/2\mathbb{Z}$ -vector spaces $H^n(X; \mathbb{Z}/2\mathbb{Z}) \rightarrow B$ carrying S_X onto S . The pair (B, S) is *realizable* if it is realized by a finite n -dimensional CW-complex.

Lemma 11.2. *Any pair $(B, S \subset B \setminus \{0\})$ where B is a finite-dimensional $\mathbb{Z}/2\mathbb{Z}$ -vector space and S generates B as a vector space, is realizable.*

Proof. We define two moves M_1, M_2 transforming S . The move M_1 adds to S a new element of the form $a + b$ for some distinct $a, b \in S$. The move M_2 takes two distinct elements a, b of S , deletes a from S , and adds $a + b$ to S . Note that if $a + b \in S$ before the application of the moves, then M_1 keeps S as is and M_2 deletes a from S . The lemma is a direct consequence of the following three claims:

- (a) If S is a basis of B , then the pair (B, S) is realizable;
- (b) For any set $S \subset B \setminus \{0\}$ generating B as a vector space, there is a sequence of moves M_1, M_2 transforming a basis of B into S ;
- (c) If a pair $(B, S \subset B \setminus \{0\})$ is realizable and a set $S' \subset B$ is obtained from S by one of the moves M_1, M_2 , then $S' \subset B \setminus \{0\}$ and the pair (B, S') is realizable.

Claim (a) is obvious: if S is a basis of B , then the pair (B, S) is realized by a wedge of n -spheres. We prove Claim (b). Since the set S generates B , it contains a basis S_0 of B . Each vector $s \in B$ expands uniquely as a sum of vectors of S_0 , and we call the number of these vectors the *size* of b . Using the moves M_1, M_2 , we can consecutively add the vectors of $S \setminus S_0$ to S_0 and thus transform S_0 into S . For example, if $s \in S \setminus S_0$ expands as $s = s_1 + s_2 + s_3$ with $s_1, s_2, s_3 \in S_0$, then we first apply M_1 adding $s_1 + s_2$ to S_0 and then apply M_2 removing $s_1 + s_2$ and adding $s = s_1 + s_2 + s_3$ instead. To avoid counteractions between the moves, we first add to S_0 the vectors of $S \setminus S_0$ of the maximal size, then of the maximal size minus 1, of the maximal size minus 2, etc.

We now prove Claim (c). Suppose that the pair (B, S) is realized by a finite n -dimensional CW-complex X . To simplify notation, we identify B with $H^n(X; \mathbb{Z}/2\mathbb{Z})$ and S with S_X . Pick any distinct $a, b \in S$. Then $a = X_e$ and $b = X_f$ where e and f are two distinct (open) n -cells of X . Consider their closures $\bar{e} \supset e$ and $\bar{f} \supset f$ in X . Subdividing if necessary the $(n-1)$ -skeleton $X^{(n-1)}$ of X we can assume that X has at least one 0-cell x_e lying in $\bar{e} \setminus e$ and at least one 0-cell x_f lying in $\bar{f} \setminus f$. Consider an n -ball D with the structure of a CW-complex formed by a 0-cell $x \in \partial D$, an $(n-1)$ -cell $\partial D \setminus \{x\}$, and an n -cell $\text{Int}(D) = D \setminus \partial D$. Pick an embedding $i : D \hookrightarrow \bar{e}$ such that $i(x) = x_e$ and $i(D \setminus \{x\}) \subset e$. Similarly, pick an embedding $j : D \hookrightarrow \bar{f}$ such that $j(x) = x_f$ and $j(D \setminus \{x\}) \subset f$. We form a quotient space Y of X by identifying $i(d) = j(d)$ for all $d \in D$. The space Y is a finite CW-complex whose $(n-1)$ -skeleton is obtained from the disjoint union $X^{(n-1)} \amalg \partial D$ by the identification $x_e = x_f = x$. The n -cells of Y are the images under the projection $X \rightarrow Y$ of the n -cells of X distinct from e, f together with the n -cells

$$e' = e \setminus i(D), \quad f' = f \setminus j(D), \quad g = i(\text{Int } D) = j(\text{Int } D).$$

Considered up to homotopy equivalence, Y is obtained from X by adjoining an arc from x_e to x_f . Since $n \geq 2$, the projection $X \rightarrow Y$ induces an additive isomorphism $H_n(X; \mathbb{Z}/2\mathbb{Z}) \rightarrow H_n(Y; \mathbb{Z}/2\mathbb{Z})$. Identifying these groups along this isomorphism, we obtain that $Y_c = X_c$ for all n -cells c of X distinct from e, f and

$$Y_{e'} = X_e = a, \quad Y_{f'} = X_f = b, \quad Y_g = a + b.$$

This shows that S_Y is obtained from $S = S_X$ by the move M_1 . This proves the part of Claim (c) concerning M_1 .

The part of Claim (c) concerning M_2 is proved similarly using the same e, f, D, i as above but a different map $j : D \rightarrow \bar{f}$. Namely, observe that the CW-complex X can be obtained from the CW-complex $X \setminus f$ by attaching the n -ball D along a map $\varphi : \partial D \rightarrow X \setminus f$. We let j be the composition of the inclusion D into $D \amalg (X \setminus f)$ with the projection of the latter space into X . Then $j|_{\partial D} = \varphi$ and j restricts to a homeomorphism $\text{Int } D \approx f$. We form a CW-complex Y by identifying $i(d) = j(d)$ for all $d \in D$. Observe that the n -cells of Y are the images under the projection $X \rightarrow Y$ of the n -cells of X distinct from e together with the n -cell $e' = e \setminus i(D)$. The projection $X \rightarrow Y$ induces an additive isomorphism $H_n(X; \mathbb{Z}/2\mathbb{Z}) \rightarrow H_n(Y; \mathbb{Z}/2\mathbb{Z})$,

and identifying these groups we obtain that $Y_c = X_c$ for all n -cells c of X distinct from e and $Y_{e'} = a + b$. Thus, S_Y is obtained from $S = S_X$ by the move M_2 . This completes the proof of Claim (c) and of the lemma. \square

11.4. Proof of Theorem 11.1. Given a plain additive poset A , Theorem 8.1 implies that the partial order in A is determined as in Theorem 3.2 by a set $S \subset A^*$ such that $\cap_{s \in S} \text{Ker } s = 0$. Eliminating if necessary the zero vector from S , we can assume that $S \subset A^* \setminus \{0\}$. Since the additive poset A is plain, its underlying $\mathbb{Z}/2\mathbb{Z}$ -vector space is finite-dimensional. Therefore, the equality $\cap_{s \in S} \text{Ker } s = 0$ implies that S generates A^* . By Lemma 11.2, there is a finite n -dimensional CW-complex X and an isomorphism $H^n(X; \mathbb{Z}/2\mathbb{Z}) \rightarrow A^*$ carrying S_X onto S . The dual isomorphism $A \rightarrow H_n(X; \mathbb{Z}/2\mathbb{Z})$ is an isomorphism of the additive poset A onto the homological additive poset of X .

12. HOMOLOGICAL ADDITIVE POSETS OF GRAPHS

12.1. Graphs and atoms. By a *graph*, we mean a 1-dimensional CW-complex. A graph Γ is obtained by attaching copies of the segment $[-1, 1]$ to a discrete set of points. The points of the latter set are called vertices of Γ and the copies of $[-1, 1]$ in question are called edges of Γ . Each edge is glued to two (possibly, coinciding) vertices of Γ called the endpoints of this edge. A 1-cycle in Γ is a finite set of edges such that every vertex of Γ is incident to an even number of edges in this set (counting with multiplicities). Every element of the group $H = H_1(\Gamma; \mathbb{Z}/2\mathbb{Z})$ is represented by a unique 1-cycle. As in Section 10, we turn H into an additive poset by providing it with the partial order $a \leq b$ if the 1-cycle representing $a \in H$ is contained in the 1-cycle representing $b \in H$. As we know, this partial order is invariant under subdivisions of Γ and turns H into an additive poset, the homological additive poset of Γ .

Lemma 12.1. *Let Γ be a graph. For any nonzero homology class $a \in H = H_1(\Gamma; \mathbb{Z}/2\mathbb{Z})$, there is a nonzero homology class $b \in H$ such that $b \leq a$ and b is represented by an embedding of a circle into Γ .*

Proof. Let E be a 1-cycle in Γ representing a . If E contains a loop, that is an edge with coinciding endpoints, then this loop determines an embedding $S^1 \hookrightarrow \Gamma$ representing a nonzero homology class $b \in H$ such that $b \leq a$. Suppose that E contains no loops. Pick any edge $e_1 \in E$ with endpoints x_1, x_2 . Since E is a 1-cycle, the vertex x_2 is incident to an edge $e_2 \in E \setminus \{e_1\}$. Let x_3 be the endpoint of e_2 distinct from x_2 . Since E is a 1-cycle, the vertex x_3 is incident to an edge $e_3 \in E \setminus \{e_2\}$. Repeating this argument, we construct a sequence of edges $e_1, e_2, e_3, \dots \in E$ such that any two consecutive edges e_{i-1}, e_i are distinct and share a vertex x_i . Since E is a finite set, there must be indices $k < l$ such that $x_k = x_l$. We take such a pair (k, l) with the smallest possible difference $l - k$. Then the edges $e_k, e_{k+1}, \dots, e_{l-1}$ are distinct and have no common vertices except that e_{i-1}, e_i share a vertex x_i for all i and e_k and e_{l-1} share a vertex $x_k = x_l$. Then the edges $e_k, e_{k+1}, \dots, e_{l-1}$ form an embedded circle in Γ . This circle represents a nonzero homology class $b \in H$ such that $b \leq a$. \square

Theorem 12.2. *Let Γ be a graph. A homology class $a \in H_1(\Gamma; \mathbb{Z}/2\mathbb{Z})$ is represented by an embedding of a circle into Γ if and only if a is an atom.*

Proof. If a is represented by an embedding $S^1 \hookrightarrow \Gamma$, then a is an atom by Theorem 10.2. Conversely, if a is an atom, then by Lemma 12.1, there is a nonzero homology class $b \in H_1(\Gamma; \mathbb{Z}/2\mathbb{Z})$ such that $b \leq a$ and b is represented by an embedding $S^1 \hookrightarrow \Gamma$. Since a is an atom, we must have $a = b$, so that a is represented by an embedding $S^1 \hookrightarrow \Gamma$. \square

12.2. Geometric tiles. To extend Theorem 12.2 to tiles, we define a family of graphs called geometric tiles. Recall first that a *wedge* of two graphs Γ_1 and Γ_2 is an arbitrary graph obtained from the disjoint union $\Gamma_1 \amalg \Gamma_2$ by identifying a vertex of Γ_1 with a vertex of Γ_2 . The wedge may depend on the choice of the vertices used in the construction.

A *geometric tile of weight 1* is a graph whose underlying topological space is homeomorphic to a circle. Suppose that geometric tiles of weights $1, 2, \dots, n$ are already defined for some integer $n \geq 1$. A *geometric tile of weight $n + 1$* is a graph which is either a wedge of a geometric tile of weight n and a geometric tile of weight 1 or a disjoint union of two geometric tiles such that the sum of their weights is equal to $n + 1$. For example, a disjoint union of $n \geq 1$ geometric tiles of weight 1 is a geometric tile of weight n . An induction on the weight $w(T) \geq 1$ of a geometric tile T shows that T expands as a union of $w(T)$ embedded circles which may meet only in vertices of T . Moreover, T does not contain embedded circles other than those in this expansion.

Given a graph Γ , any homology class $a \in H_1(\Gamma; \mathbb{Z}/2\mathbb{Z})$ is represented by a unique 1-cycle. The edges of Γ belonging to this 1-cycle and their endpoints form a graph $\Gamma_a \subset \Gamma$ called the *support* of a .

Theorem 12.3. *Let Γ be a graph. An element of $H = H_1(\Gamma; \mathbb{Z}/2\mathbb{Z})$ is a tile if and only if its support is a geometric tile.*

Proof. If the support Γ_a of $a \in H$ is a geometric tile, then a expands as a sum of $w(\Gamma_a) \geq 1$ homology classes represented by embedded circles which meet only in vertices of Γ . By Theorem 12.2, these homology classes are atoms. Clearly, they are pairwise independent. Moreover, any atom smaller than or equal to a is represented by an embedded circle in Γ_a , and so is equal to one of the atoms in our expansion of a . Thus, a is a tile.

Conversely, suppose that $a \in H$ is a tile. By Theorem 6.4, all atoms a_1, \dots, a_n of A_a are pairwise independent and their sum is equal to a . By Theorem 12.2, the atoms a_1, \dots, a_n are represented by embedded circles, respectively, $S_1, \dots, S_n \subset \Gamma$. Since a_1, \dots, a_n are pairwise independent, these circles meet only in vertices of Γ . We prove that Γ_a is a geometric tile by induction on n . If $n = 1$, then $a = a_1$ and $\Gamma_a = S_1$ is a circle. We explain the induction step. Set $b = a_1 + \dots + a_{n-1}$. Note that $a_i \leq a$ for all i and therefore $b \leq a$. Since a is a tile, so is b . By the induction assumption, the support $\Gamma_b = S_1 \cup \dots \cup S_{n-1}$ of b is a geometric tile. We claim that the circle S_n cannot meet a connected component of Γ_b in more than one point. This easily implies that $\Gamma_a = \Gamma_b \cup S_n$ is a geometric tile. To prove the claim, suppose that S_n meets a component Δ of Γ_b in two or more vertices. Then there is an arc $\alpha \subset S_n$ meeting Δ precisely in its endpoints. Pick an embedded path β connecting these endpoints in Δ . The union $\alpha \cup \beta$ is an embedded circle in Γ_a distinct from the circles S_1, \dots, S_n . This is, however, impossible because all atoms of A_a belong to the list a_1, \dots, a_n . This proves the claim above and completes the proof of the theorem. \square

12.3. Finite graphs. A graph is *finite* if its sets of vertices and edges are finite. For a finite graph Γ , the additive poset $H = H_1(\Gamma; \mathbb{Z}/2\mathbb{Z})$ is plain. By (10.1.1), the complexity of H is smaller than or equal to the number of edges of Γ . By Theorem 6.1, every element of H expands in a canonical way as a sum of atoms. Other results and definitions of Sections 6–12 also apply to H ; we leave the details to the reader.

12.4. Remarks. 1. The homological additive poset of a graph is functorial with respect to graph inclusions: if a graph Γ is contained in a graph Γ' , then the inclusion homomorphism

$$\text{in}_{\Gamma, \Gamma'} : H_1(\Gamma; \mathbb{Z}/2\mathbb{Z}) \rightarrow H_1(\Gamma'; \mathbb{Z}/2\mathbb{Z})$$

is an embedding of additive posets which carries atoms to atoms and tiles to tiles. A similar result holds for CW-complexes of any dimension.

2. Given a graph Γ and atoms $b, c \in H = H_1(\Gamma; \mathbb{Z}/2\mathbb{Z})$, one may ask whether the circles in Γ representing b and c are disjoint. A necessary condition is that b and c are independent and $b + c$ is a tile. However, this condition is insufficient because it leaves open the possibility that the circles representing b and c meet in a vertex of Γ . Consider the map $\chi : H \rightarrow \mathbb{Z}$ carrying a homology class to the Euler characteristic of its support. The circles in Γ representing the atoms $b, c \in H$ are disjoint if and only if b, c are independent, $b + c$ is a tile, and $\chi(b + c) = 0$. An equivalent condition may be formulated in terms of graphs containing Γ as a subgraph. Suppose that $b, c \in H$ are independent atoms whose sum is a tile. The circles in Γ representing b, c are disjoint if and only if the following holds: for any graph $\Gamma' \supset \Gamma$, and any atom $d \in H' = H_1(\Gamma'; \mathbb{Z}/2\mathbb{Z})$ independent from $b' = \text{in}_{\Gamma, \Gamma'}(b) \in H'$ and $c' = \text{in}_{\Gamma, \Gamma'}(c) \in H'$, if $d + b'$ and $d + c'$ are tiles, then $d + b' + c'$ is a tile.

3. A homology class a of a graph Γ is represented by a finite disjoint union of circles embedded in Γ if and only if a is a tile and any two atoms smaller than or equal to a are represented by disjoint circles in Γ .

13. REALIZATION OF ADDITIVE POSETS BY GRAPHS

The following theorem shows that some plain additive posets are not realizable by graphs.

Theorem 13.1. *Let m be a positive integer and $A = A_m$ be an m -dimensional $\mathbb{Z}/2\mathbb{Z}$ -vector space with the trivial partial order. For $m \leq 4$, there is a graph whose homological additive poset is isomorphic to A . For $m \geq 5$, there is no such graph.*

Proof. The additive poset A_1 is isomorphic to the homological additive poset of the graph having one vertex and one edge (which is a loop). The additive poset A_2 is isomorphic to the homological additive poset of the graph formed by two vertices and three connecting them edges. The additive poset A_3 is isomorphic to the homological additive poset of the complete graph on four vertices. The additive poset A_4 is isomorphic to the homological additive poset of the graph $K_{3,3}$ having six vertices three of which are connected to each of the other three by a single edge. In all these cases, each nonempty 1-cycle of the graph is an embedded circle and so represents an atom in H_1 . Therefore the partial order in H_1 is trivial.

It remains to prove that there is no graph Γ such that the partial order in the homological additive poset $H = H_1(\Gamma; \mathbb{Z}/2\mathbb{Z})$ is trivial and $\dim(H) \geq 5$. Suppose

that there is such a graph Γ . Pick an arbitrary nonzero $a \in H$. Since the partial order in H is trivial, a is an atom. By Theorem 12.2, the support Γ_a of a is an embedded circle in Γ . Consider the graph Δ obtained from Γ by removing the interiors of all edges of Γ_a but keeping the vertices of Γ_a . We claim that $H_1(\Delta; \mathbb{Z}/2\mathbb{Z}) = 0$. Otherwise, there would exist a nonzero homology class $b \in H$ with support in Δ . Then $a \leq a + b$ which contradicts the assumption that the partial order in H is trivial. Thus, $H_1(\Delta; \mathbb{Z}/2\mathbb{Z}) = 0$. Therefore, all connected components of Δ are trees. We will use the following obvious property of trees: any two distinct points u, v of a tree are related by a path \overline{uv} in this tree having no self-intersections.

Fix from now on an orientation of the circle Γ_a . For any distinct points $u, v \in \Gamma_a$ we let \widehat{uv} be the arc on Γ_a leading from u to v . If the points u, v lie on the same component T of Δ , then we have an embedded circle

$$S_{uv} = \widehat{uv} \cup \overline{uv} \subset \Gamma_a \cup T \subset \Gamma_a \cup \Delta.$$

Let $\{T_i\}_i$ be the components of Δ meeting Γ_a . Let $k_i \geq 1$ be the number of common vertices of Γ_a and T_i . Clearly,

$$\dim(H) = 1 + \sum_i (k_i - 1).$$

Since $\dim(H) \geq 5$, at least one of the following conditions hold:

- (a) Γ_a meets a component T of Δ in ≥ 5 vertices;
- (b) Γ_a meets a component T of Δ in ≥ 3 vertices and another component T' of Δ in ≥ 2 vertices;
- (c) Γ_a meets four distinct components T_1, T_2, T_3, T_4 of Δ in two vertices each.

In all cases, we construct two non-empty 1-cycles in $\Gamma_a \cup \Delta$ meeting only in vertices. The homology classes $b, c \in H$ of these 1-cycles are nonzero and satisfy $b \leq b + c$. This contradicts the assumption that the partial order in H is trivial.

Case (a). Let u, v, w, x, y be five distinct points of $\Gamma_a \cap T$ enumerated in the cyclic order on Γ_a . If the paths \overline{uv} and \overline{xy} meet only in vertices, then so do the circles S_{uv} and S_{xy} . Suppose that the paths \overline{uv} and \overline{xy} have a common edge. The complement of the interior of this edge in T is a union of two disjoint subtrees of T . Clearly, the points u, v lie in different subtrees and so do x, y . If v, x lie in the same subtree, then so do y, u and $S_{vx} \cap S_{yu} = \emptyset$. Suppose that v, y lie in one subtree and u, x lie in the other subtree. If w lies in the same subtree as v , then $S_{vw} \cap S_{xu} = \emptyset$. If w lies in the same subtree as x , then $S_{wx} \cap S_{yv} = \emptyset$.

Case (b). Let u, v, w be distinct points of $\Gamma_a \cap T$ and let x, y be distinct points of $\Gamma_a \cap T'$. The points u, v, w split Γ_a into three arcs, and at least one of them, say, \widehat{uv} contains neither x nor y . Consider the unique arc $\alpha \subset \Gamma_a \setminus \widehat{uv}$ with endpoints x, y . Then the following two circles are disjoint:

$$S_{uv} \subset \Gamma_a \cup T \quad \text{and} \quad \alpha \cup \overline{xy} \subset \Gamma_a \cup T'.$$

Case (c). For $i = 1, \dots, 4$, let u_i, v_i be the points of $\Gamma_a \cap T_i$. Note that if there are disjoint arcs $\alpha, \beta \subset \Gamma_a$ such that $\partial\alpha = \{u_i, v_i\}$ and $\partial\beta = \{u_j, v_j\}$ with distinct $i, j \in \{1, 2, 3, 4\}$, then the following circles are disjoint:

$$\alpha \cup \overline{u_i v_i} \subset \Gamma_a \cup T_i \quad \text{and} \quad \beta \cup \overline{u_j v_j} \subset \Gamma_a \cup T_j.$$

Assuming that there are no such arcs α, β , we observe that up to the choice of enumeration of the components T_1, T_2, T_3, T_4 and permutation of the symbols u_i, v_i ,

the cyclic order of the points $\{u_i, v_i\}_i$ on the circle Γ_a is

$$u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4.$$

Then the following two unions of arcs represent disjoint 1-cycles:

$$\widehat{u_1 u_2} \cup \overline{u_2 v_2} \cup \widehat{v_2 v_1} \cup \overline{v_1 u_1} \subset \Gamma_a \cup T_1 \cup T_2$$

and

$$\widehat{u_3 u_4} \cup \overline{u_4 v_4} \cup \widehat{v_4 v_3} \cup \overline{v_3 u_3} \subset \Gamma_a \cup T_3 \cup T_4.$$

□

14. OPEN QUESTIONS

We formulate several open questions concerning additive posets.

1. Are all finite additive posets plain? By Theorem 8.1, this question may be restated as follows: is the set of order-preserving linear functionals on each finite additive poset separating?

2. A *rank* of an additive poset A is a map $r : A \rightarrow \{0, 1, 2, \dots\}$ such that $r^{-1}(0) = 0$, for any independent $a, b \in A$ we have $r(a+b) = r(a) + r(b)$ and for any finite nonempty set $K \subset A$, the integer

$$\sum_{\emptyset \neq J \subset K} (-1)^{|J|+1} r\left(\sum_{a \in J} a\right)$$

is nonnegative and divisible by $2^{|K|-1}$. An injective morphism of additive posets $\varphi : A \rightarrow 2_f^I$, where I is a set, determines a rank of A by $r(a) = |\varphi(a)|$ for all $a \in A$. Thus, every plain additive poset has a rank. Does every finite additive poset have a rank? Are all finite additive posets having a rank plain? Is every rank of an additive poset induced by an injective morphism into an additive powerset?

3. Is the complexity of any finite additive poset A with trivial partial order equal to $m(m+1)/2$ where $m = \dim(A)$?

4. Given additive posets A and B , the additive poset $A \oplus B$ is the direct sum of the underlying abelian groups of A and B with the cartesian partial order: $(a, b) \leq (a', b')$ if $a \leq a'$ and $b \leq b'$ for any $a, a' \in A$ and $b, b' \in B$. It is easy to check that if A and B are plain, then so is $A \oplus B$ and $c(A \oplus B) \leq c(A) + c(B)$. Is this inequality an equality?

5. For any $m \geq 1$, an m -antichain in an additive poset A is an antichain in A whose elements generate a vector subspace of A of dimension $\leq m$. The m -width $w_m(A)$ of A is the maximal number of elements in an m -antichain in A . Clearly,

$$1 = w_1(A) \leq w_2(A) \leq \dots \leq w_{\dim(A)}(A) = w(A).$$

How to compute the m -width of finite additive powersets? Such a computation may help in finding new estimates of the complexity of additive posets.

6. If P is the underlying poset of a finite additive poset, then $\log_2(|P|) \in \mathbb{Z}$ and

$$h(P) \leq \log_2(|P|) \leq w(P),$$

see Theorem 7.1. For example, for an integer $n \geq 1$, the set $\{0, 1, \dots, n\}$ with partial order $0 \leq 1 \leq \dots \leq n$ has the height n . Since the inequality $n \leq \log_2(n+1)$ holds only for $n = 1$, the poset $\{0, 1, \dots, n\}$ with $n \geq 2$ does not underlie an additive poset. Are there further algebraic conditions on the posets underlying finite additive posets?

7. By Theorem 13.1, finite additive posets of dimension ≥ 5 with trivial partial order are not isomorphic to the homological additive posets of graphs. Can one describe algebraically the class of homological additive posets of graphs?

We note a related question. If A is the homological additive poset of a finite connected graph $\Gamma \neq S^1$, then $c(A) \leq 3 \dim(A) - 3$. Indeed, without loss of generality we can assume that all vertices of Γ have valency ≥ 3 . Let V be the number of vertices and E be the number of edges of Γ . Then

$$\dim(A) - 1 = -\chi(\Gamma) = E - V \geq E - 2E/3 = E/3.$$

So, $E \leq 3 \dim(A) - 3$. Since $c(A) \leq E$, we have $c(A) \leq 3 \dim(A) - 3$. Are there plain additive posets which do not satisfy this inequality?

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